Synchronized spatiotemporal chaos and spatiotemporal on-off intermittency in a nonlinear ring cavity

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We numerically study the transverse Ikeda system describing a ring cavity with a saturable absorber nonlinearity which is driven by a coherent input field. The Gaussian shape of the input beam favors the solutions to have space inversion symmetry. These solutions bifurcate into states of synchronized spatiotemporal chaos where synchronization refers to the identical dynamics of the two beam halves. We demonstrate that the breakdown of this synchronized chaotic state is caused by on-off intermittency. The qualitative and quantitative features of the transition are studied. In particular, the scaling behavior of the transition in the space domain is investigated. The results display a qualitative similarity with the scaling behavior of spatiotemporal intermittency. [S1063-651X(96)06809-2]

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I. INTRODUCTION

The complex behavior of spatially extended systems has been the subject of considerable research effort in various scientific disciplines [1]. Various bifurcation routes generating spatiotemporal chaos such as defect-mediated turbulence [2], spatiotemporal intermittency [3,4], and chaotic itinerancy [5–8] have been investigated. Furthermore, it has been shown that in relatively small systems spatially simple solutions which are chaotic in time evolve towards spatiotemporal chaos by a spontaneous breaking of spatial symmetries [9–11]. This situation applies, e.g., for systems displaying synchronized spatiotemporal chaos. This dynamic state appears when a spatially extended system can be divided into two identical subsystems which are coupled to each other. Under certain circumstances, the subsystems will synchronize even if the dynamics in every subsystem is spatiotemporally complex [12]. In this contribution we demonstrate that in the transverse Ikeda system synchronized spatiotemporal chaotic states occur and that they bifurcate towards states of higher spatiotemporal disorder via on-off intermittency.

On-off intermittency has attracted considerable interest as a mechanism for intermittent bursting during the last few years [13,14]. Different from the known Pomeau-Manneville (PM) and crisis-induced intermittency, its basic mechanism is the action of a dynamical variable as a time-dependent driving parameter for a second variable y. If this driving corresponds to a modulation through a bifurcation point, y shows intermittent behavior. Hereby, the driving variable can either be a stochastic or a chaotic process. If chaotic processes as driving parameters are concerned, the necessary condition for on-off intermittency is that the phase space of the system can be divided in two hyperplanes where in hyperplane I the chaotic process is generated and in hyperplane II $y_1=\cdots=y_k=0$ holds. Consequently, the system displays the following features: for a bifurcation parameter $a<e$, where e is the threshold for intermittency, the signal y decays to zero. For $a\geqslant e$, intermittent behavior of y is observed with long periods near $y=0$ (laminar phases) and sudden intermittent bursts with large amplitudes of y. In the intermittent regime, two scaling laws have been derived [15]: (a) the probability distribution $P(L)$ of the duration of the laminar phases L at a fixed parameter value obeys a power-law scaling of the form $P(L)\sim L^{-3/2}$, (b) in dependence on the parameter value a, the mean duration of the laminar phases $\langle L\rangle$ scales as $\langle L\rangle\sim(a-e)^{-1}$. These results were analytically derived for stochastic driving processes. Although no proof is available for chaotic driving processes, numerical investigations showed the same scaling behavior in this case. While on-off intermittency can thus be confused with PM intermittency of type III if only its characteristic scaling behavior is considered, the qualitative features of on-off intermittency distinguish it clearly from the former.

An early example of on-off intermittency has been reported for the breakdown of synchronized chaos in coupled logistic maps [16]. More recent studies have witnessed the widespread occurrence of this instability: systems where on-off intermittency has been studied theoretically comprise coupled ordinary differential equations, a model for Zeeman lasers [17] and stochastically driven map lattices [18,19]. In this context, on-off intermittency has also been called ''modulational spatiotemporal intermittency'' [20] and has been related to the growth of interfaces in random media. Experimental evidence for on-off intermittency has been given in nonlinear electronic circuits [14] and in spin-wave instabilities [21].

The systems where on-off intermittency has been observed until today are thus low-dimensional systems or distributed systems subject to stochastic driving. We will argue that in spatially extended systems on-off intermittency occurs spontaneously, i.e., without the need for stochastic driving when certain conditions hold. The reason for this behavior is that in spatially extended systems with a moderate aspect ratio, solutions occur which are highly disordered in time but which retain some spatial symmetries. Further bi-

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furcations which lead to states with higher spatiotemporal disorder then necessarily break these symmetries. Examples of such bifurcations have been given for hydrodynamical [10] and nonlinear optical systems [11,22]. For suitable coordinates y_i which are linked to the nonbroken symmetries, $y_i=0$ holds, thus fulfilling the necessary condition for on-off intermittency as discussed above.

In the case of synchronized motion, $y_i=0$ are simply the differences between the signals of the identical subsystems. In the model under consideration here, the identical subsystems are provided by the identical beam halves of the nonlinear optical system. This is due to the symmetry of the Gaussian input beam which favors solutions which are inversion symmetric with respect to the beam center. These solutions retain their symmetry even in the chaotic regime, causing synchronized chaotic motion of the signal at points with the same distance from the beam center. This synchronized motion breaks down in an intermittent process, leading to more complex spatiotemporal behavior. We show that this breakdown is caused by on-off intermittency and investigate its qualitative and quantitative features both in the time and space domain.

II. MODEL

The model system is a generalization of the well-known Ikeda model of optical bistability [23] which has proven to be of great value for the investigation of nonlinear phenomena due to its simplicity and its rich bifurcation behavior [24,25]. It describes a ring cavity of length $\mathcal{L}$ which contains a medium of length L_{med} of homogeneously broadened two-level atoms. We restrict our study to one transverse dimension and assume that the polarization and the population inversion of the two-level atoms can be adiabatically eliminated. The light propagation through the medium is then described by the following wave equation [23]:

$$\left(2i\frac{\partial}{\partial z}+\frac{1}{k}\frac{\partial^2}{\partial x^2}\right)E_n(x,z)+i\alpha(1+i\Delta)\frac{1}{1+4E_nE_n^*}E_n(x,z)=0. \tag{1}$$

The boundary condition imposed by the ring cavity is

$$E_n(x,0)=\sqrt{T}A(x)+R\,\exp(ik\mathcal{L})E_{n-1}(x,L_{\mathrm{med}}). \tag{2}$$

In the wave equation, α denotes the unsaturated absorption coefficient, Δ the atomic detuning, and k the wave number of the input beam. The index n labels the electric field E in the nth resonator pass; z is the longitudinal and x the transverse coordinate. In the boundary condition, $A(x)$ denotes the Gaussian input beam $A(x)=A\,\exp(-x^2/w_{1/2}^2)$ where $w_{1/2}$ is the full width at half maximum. The results reported in the following are obtained for parameter values

$$\alpha L_{\mathrm{med}}\Delta=-10,\quad \alpha L_{\mathrm{med}}=0.01,\quad T=0.1,\quad k\mathcal{L}=5.88,\quad k=100, \tag{3}$$

which implies a defocusing nonlinearity. Similar results are obtained for large regions of the parameter space. We numerically solved the model equations using both a split-step method and a Crank-Nicholson finite difference scheme yielding almost perfect agreement with each other.

III. SYNCHRONIZED SPATIOTEMPORAL CHAOS AND ON-OFF INTERMITTENCY

Equations (1) and (2) were extensively analyzed as a simplified model for passive optical systems [23,26–28]. It was analytically shown that modulational instabilities dominate the linear stability of the system and that in the focusing regime, i.e., $\Delta>0$, solitonlike structures appear. Numerical solutions of the model displayed a rich variety of bifurcations into chaos. Later work concentrated on the pattern formation in the chaotic regime [11] and showed that crisis-induced intermittency plays a major role in the development of spatiotemporal chaos [22]. In this contribution we analyze on-off intermittent bifurcations of the synchronized chaotic state and show that these bifurcations produce spatiotemporally disordered states.

The bifurcation behavior of the system is investigated by variation of the amplitude A of the input beam. For increase of A, the system undergoes first a modulational instability which leads from a fixed point with a Gaussian space structure to a period-doubled solution with periodic spatial modulation. A subsequent period doubling and two Hopf bifurcations lead to a two-band chaotic attractor.

The beam profiles in this chaotic state are symmetric with respect to the $x=0$ axis [Fig. 1(a)]. Signals at the same distance from the beam center oscillate synchronously yet chaotically in time, which means that it is a state of synchronized chaos [9,16,22,29,30]. Signals within the same beam half, however, are only weakly correlated. The dynamics within one beam half is thus spatiotemporally chaotic while it is perfectly synchronized with the dynamics of the other beam half. The occurrence of this synchronized spatiotemporal chaotic state is due to the fact that only modulational instabilities have occurred, i.e., only pairs of modes with transverse wave vectors $\vec{k}_1=-\vec{k}_2$ oscillate. This causes the solutions to have space inversion symmetry. The occurrence of traveling wave instabilities which are widespread in lasers would completely modify the situation, causing symmetry-broken solutions already in the regular regime. The synchronization of the two beam halves is caused by the diffractive coupling. Since this is a local effect, the beam halves are only indirectly coupled through the beam center. From there, information spreads symmetrically to both beam parts and acts thus as a synchronizing signal. It is thus one signal which is transported through the subsystems and synchronizes their dynamics. Similar situations have been described for spatiotemporal chaos in coupled map lattices [12] where it is possible to synchronize different subsystems by linking only a few of their dynamical variables.

As the amplitude of the input beam A is further increased, the dynamics becomes more chaotic while the strength of the linking signal being determined by the diffraction strength remains constant. Consequently, this leads to a breakdown of the synchronization at $A\sim1.0191$ with apparently no correlation between the beam halves [Fig. 1(b)]. This situation is very reminiscent of the first observation of on-off intermittency in symmetrically coupled logistic maps which is due to a formal correspondence between these two cases: two coupled maps

$$x_{n+1}^{(1)}=(1-\epsilon)f(x_n^{(1)})+\epsilon f(x_n^{(2)}), \tag{4}$$

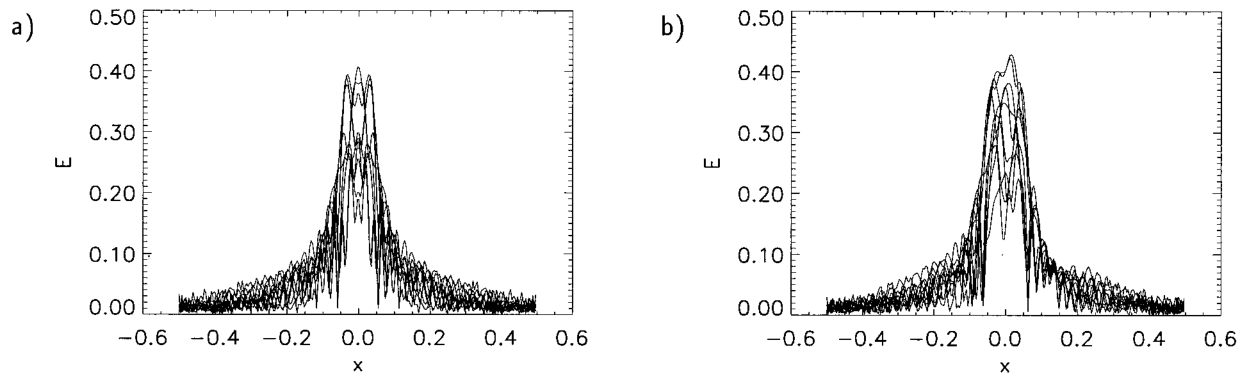

FIG. 1. (a) Ten synchronized beam profiles in the chaotic region ($A = 0.98$); (b) ten beam profiles in the chaotic region where synchronization has broken down ($A = 1.05$). In all figures the transverse coordinate x is normalized to unity and the intensity is given in dimensionless units.

$$x_{n+1}^{(2)} = (1-\epsilon) f(x_n^{(2)}) + \epsilon f(x_n^{(1)}) \tag{5}$$

can be transformed by a rotation of the coordinate system by $\pi/4$ to the variables $r = (x^{(1)} + x^{(2)})/2$ and $s = (x^{(1)} - x^{(2)})/2$:

$$r_{n+1} = f(r_n) + g(r_n, s_n), \tag{6}$$

$$s_{n+1} = h(r_n, s_n)(1-2\epsilon) s_n, \tag{7}$$

$$g(r_n, s_n) = [f(r_n + s_n) + f(r_n - s_n) - 2f(r_n)]/2, \tag{8}$$

$$h(r_n s_n) = [f(r_n + s_n) - f(r_n - s_n)]/2 s_n. \tag{9}$$

Here r_{n+1} describes the synchronized part of the signal while s_{n+1} describes its nonsymmetric parts. In this representation, an important property of Eqs. (4) and (5) is revealed: if a synchronized chaotic process exists, it is completely described by the dynamics of r_n while $s_n = 0$ holds. Then r_n acts as a chaotic driver for s_n while — to the leading order in $s_n - s_n$ does not influence the dynamics of r_n [16]. Since this is the necessary condition for the occurrence of on-off intermittency as discussed in Sec. I, ensembles of identical coupled maps displaying synchronized chaos are likely to display this bifurcation. In spatially extended systems whose solutions obey inversion symmetry this transformation can be applied pointwise for variables at points lying symmetrically to the symmetry axis. It is thus the symmetrical modes with $E(x,t) = E(-x,t)$ where x is the distance from the symmetry axis which drive the nonsymmetric ones. Spatially extended chaotic systems with inversion symmetry are therefore to the same extent likely to display on-off intermittency as are ensembles of coupled maps.

To further elucidate this point, we introduce a quantity which measures the amount of symmetry breaking of the beam profiles, namely,

$$S = \int_{-0.2}^{0.2} |E(-x) - E(x)| dx. \tag{10}$$

This quantity corresponds to s_{n+1} in the case of coupled maps. For symmetric profiles, $S = 0$ holds, $S > 0$ otherwise. A typical time series of S above the bifurcation to nonsymmetric solutions shows the characteristic qualitative feature of on-off intermittent signals: regions with $S = 0$ (laminar phases) and $s \neq 0$ (turbulent phases) are clearly distinguishable. The transitions between these phases are sharp and do not display oscillatory behavior which is characteristic for on-off intermittency (Fig. 2). To investigate the quantitative properties of the transition we calculate the distribution statistics of the laminar phases in dependence on (a) the distance of the bifurcation parameter A from the bifurcation point A_c and (b) the duration of the laminar phase at a fixed bifurcation parameter. The results of this analysis are shown in Figs. 3 and 4, respectively. The duration of the mean laminar phase $\langle L \rangle$ obeys a power-law scaling with a critical exponent of -1. The distribution $P(L)$ of the duration of the laminar phases at a fixed bifurcation parameter shows a power-law scaling with an exponent $-\frac{3}{2}$. A variation of the threshold which distinguishes between laminar and turbulent phases does not evidence a significant dependence of these scaling laws.

The scaling characteristics and the qualitative features of the transition thus coincide with the observations reported for other systems showing on-off intermittency. Examination of the spatial properties of the transition shows new interesting features. By evaluating the normalized local amount of symmetry breaking

$$s(x) = [|E(-x)| - |E(x)|]/[|E(-x)| + |E(x)|] \tag{11}$$

at different locations x we observe that at all locations the symmetry breaking bursts occur approximately at the same

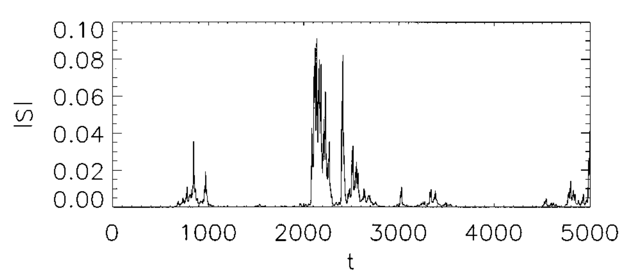

FIG. 2. Time series of the integrated symmetry breaking S at $A = 1.03$.

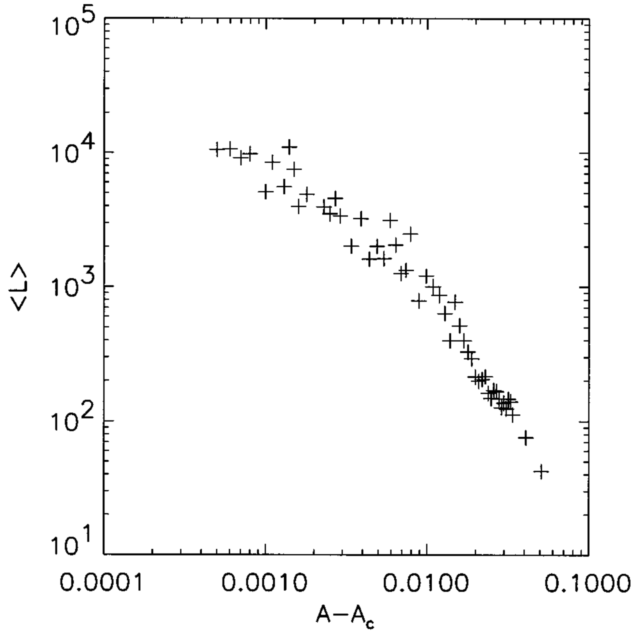

FIG. 3. Dependence of the mean duration of laminar domains from the distance from the bifurcation point $A-A_c$.

time, but with different amplitude (Fig. 5). Therefore also in the spatial domain a distribution of laminar and turbulent regions exists. This becomes evident in Fig. 6(a) where $s(x)$ is displayed for successive cavity passes in a contour representation. An irregular distribution of areas with high and low values of $s(x)$ is observed. The stripelike organization of $s(x)$ is due to the defocusing nonlinearity which transports perturbations to the outer pars of the beam. Since the influence of the linking signal is highest in the beam center and decreases in the outer parts, $s(x)$ increases with increasing x. This is demonstrated in Fig. 6(b) where $s(x)$ is time averaged and plotted as a function of x. A roughly linear increase of $s(x)$ with increasing x is revealed, while

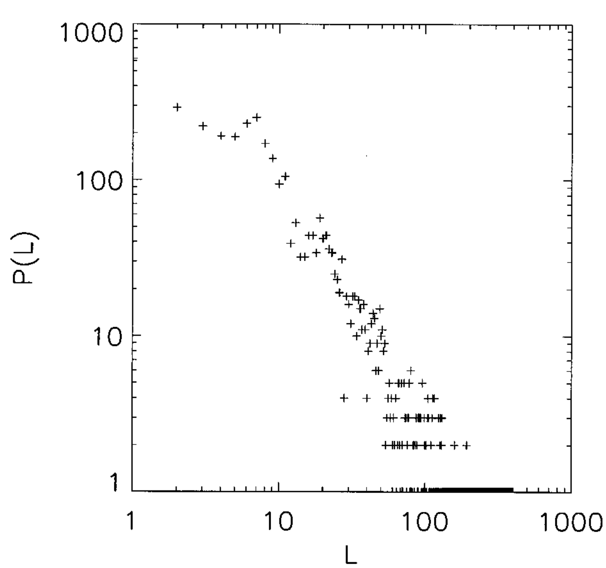

FIG. 4. Probability distribution $P(l)$ of laminar domains of duration l at a fixed parameter value ($A=1.035$).

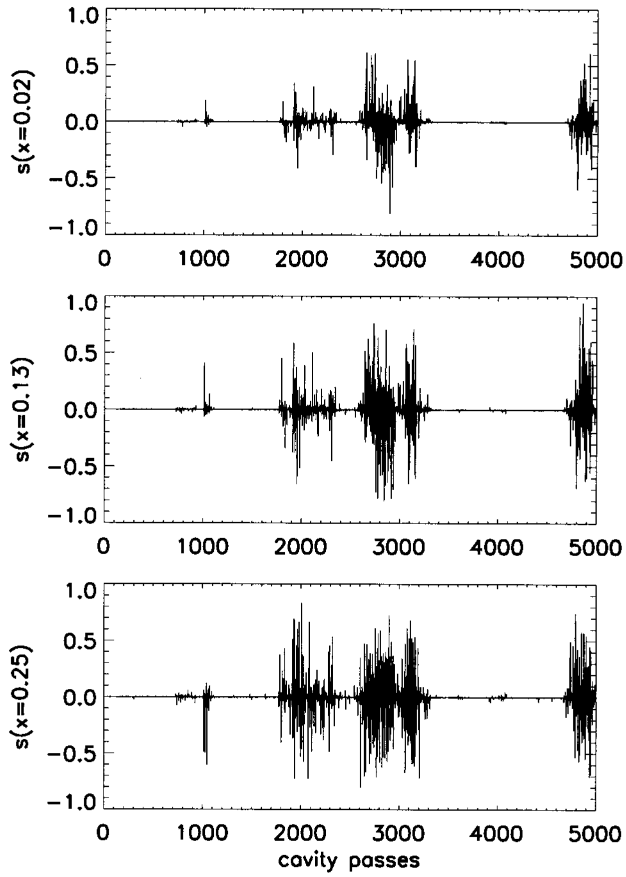

FIG. 5. Time series of the normalized local symmetry breaking $s(x)$ at different locations x ($A=1.035$).

the superimposed oscillations reflect the space frequencies of $E(x)$. By setting a cutoff for $s(x)$ as a distinguishing criterion between laminar and turbulent domains we are able to calculate the probability distribution of the laminar phases. Thereby we limit our investigation to the central part of the beam, discarding laminar phases with a spatial length of more than $L=0.2$. This is necessary because the contributions of the tail of the beam cannot be considered as significant due to the low values of the input beam in these parts. The spatial distribution of the laminar phases is computed for every beam profile and then summed over many cavity passes. Figure 7 shows the resulting distribution of laminar phases with spatial length L at a fixed parameter value. We observe a definite power-law scaling with a characteristic exponent m_0. At higher parameter values, slight deviations from a power law are observed: the slope of the curve increases for long laminar phases, causing an upper and a lower bound for the characteristic exponent. In Fig. 8 the dependence of the characteristic exponent m_0 of the power law on the distance of the parameter value from the bifurcation point $A-A_c$ is plotted. m_0 increases with the distance of the parameter value from the bifurcation point $A-A_c$ for small values of $A-A_c$ while it saturates at higher values. No clear functional relation for the dependence of m_0 on $A-A_c$ can be observed.

The observed scaling behavior shares common features with the scaling reported for spatiotemporal intermittency [3,4]. There the probability distribution of the spatial lengths

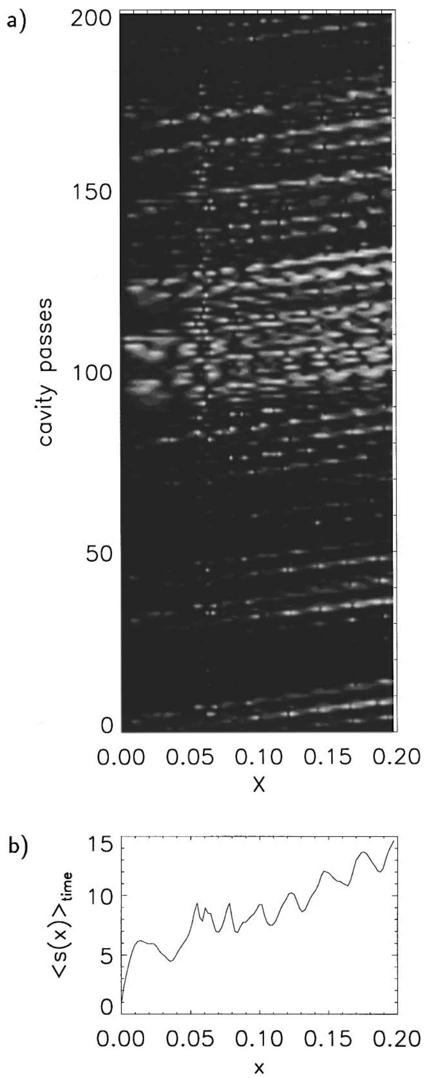

FIG. 6. (a) Gray-scale contour plot of the local symmetry breaking $s(x)$ as a function of space and time at $A=1.045$. Dark regions indicate low values of s, light regions high values. (b) $s(x)$ time averaged for 10 000 cavity passes.

of the laminar phases displays a power law at parameters near the onset of spatiotemporal intermittency and exponential decay for higher parameter values. The characteristic exponent m_0 of the latter depends quadratically on the distance from the border between these two regimes while the characteristic exponent of the power law does not change when the bifurcation parameter is varied. In the case investigated here, a power-law distribution of the laminar phases has also been observed directly above the onset of on-off intermittency. Its exponent, however, depends on the bifurcation parameter. While no clear functional relation for the dependence of m_0 on $A-A_c$ can be extracted, the deviations from a power law for long laminar phases at higher parameter values might indicate a qualitative change of the functional dependence of $P(L)$. We conclude that on-off intermittency in extended systems and spatiotemporal intermittency therefore bear strong qualitative similarity which need further investigation.

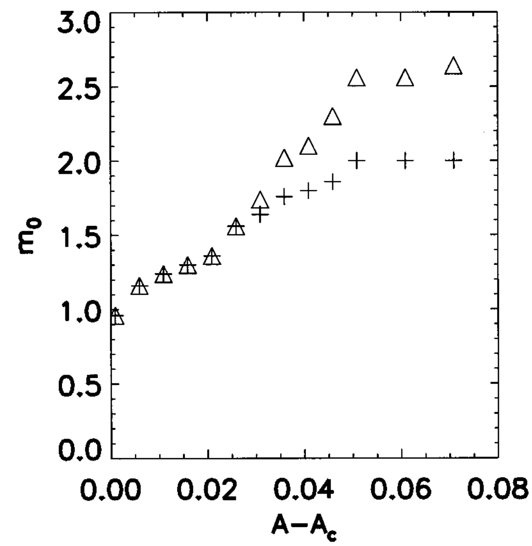

FIG. 8. The characteristic exponent m_0 of the power-law scaling as a function of the distance from the bifurcation point $A-A_c$. Plus signs mark the lower bound for m_0 while triangles denote its upper bound.

IV. CONCLUSION

The present study has revealed that the breakdown of synchronized spatiotemporal chaos in the transverse Ikeda sys-

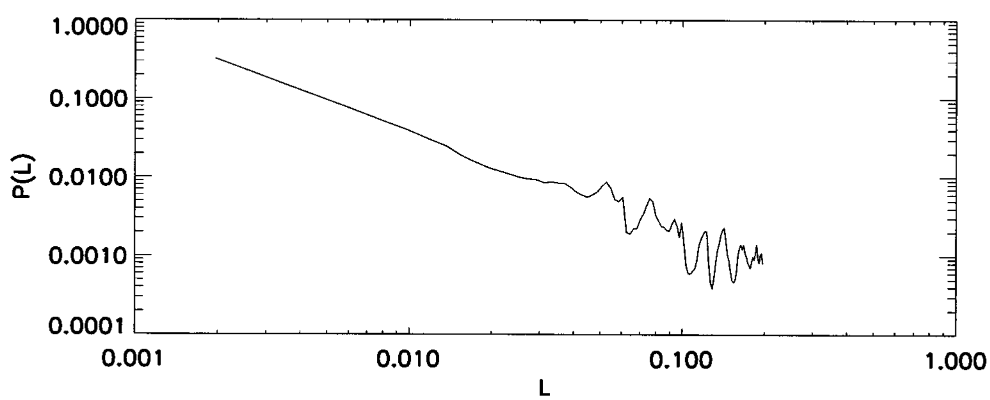

FIG. 7. Probability distribution of laminar phases with a spatial length L at a fixed parameter value ($A=1.035$).

tem is due to spatiotemporal on-off intermittency. The scaling behavior of this transition in the time domain coincides with the one analytically derived and observed in other systems displaying on-off intermittency. In the space domain the system displays a clear scaling behavior for which, however, no analytical derivation is available. Work in this direction is therefore desirable.

Since the bifurcations are observed in wide parameter regions we expect that the described effects are amenable to experimental observation. We assume that this is particularly the case for a system where the nonlinearity is provided by a liquid crystal light valve which interacts with an electronically delayed light field, thus providing the fast saturable absorber nonlinearity encountered in our model [31].

Finally, since the conditions necessary for the occurrence of on-off intermittency are very often met in systems displaying synchronized chaotic motion it is expected that systems showing synchronized spatiotemporal chaos will exhibit similar transitions.

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